Bott–Samelson manifolds, Loop groups and the Path Model

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Universität zu Köln

Motivation

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- **Littelmann path model** [1995]: Combinatorial model for Lie algebra representations

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- **e.g.:** Moment polytopes, Duistermaat-Heckmann measures, but this time symplectic

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Notation

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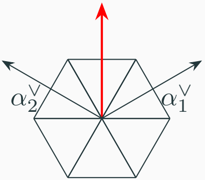

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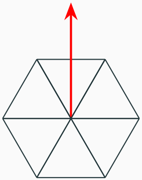

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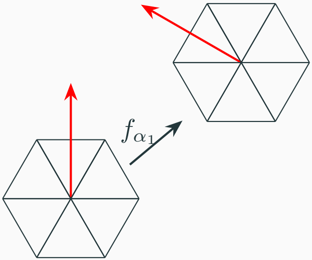

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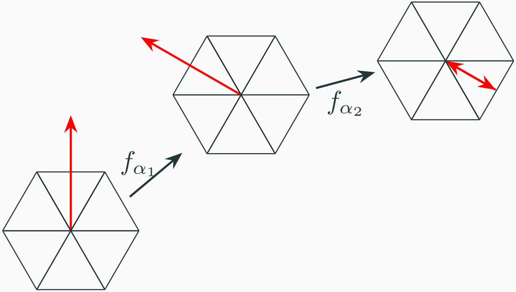

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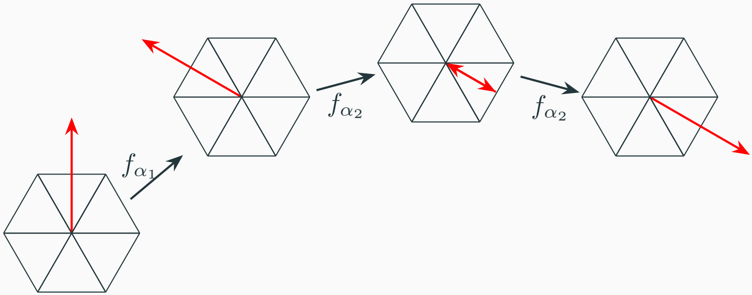

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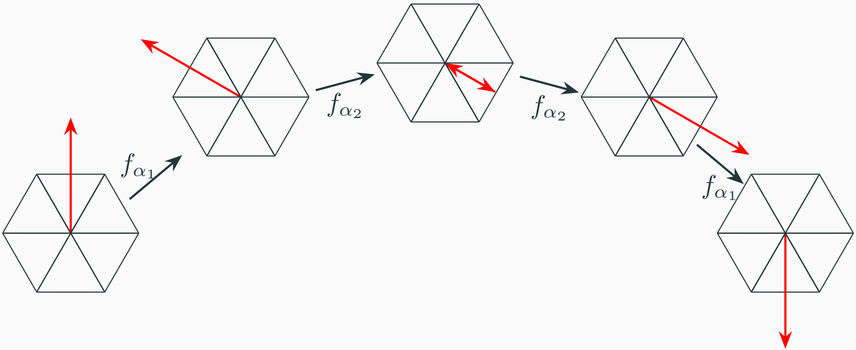

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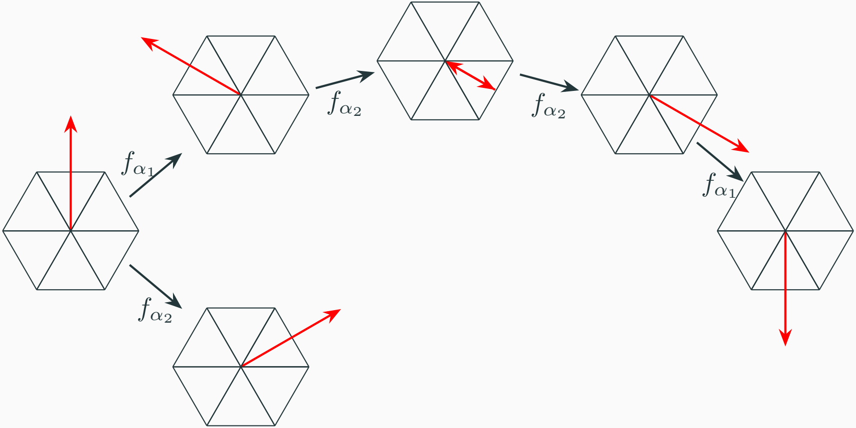

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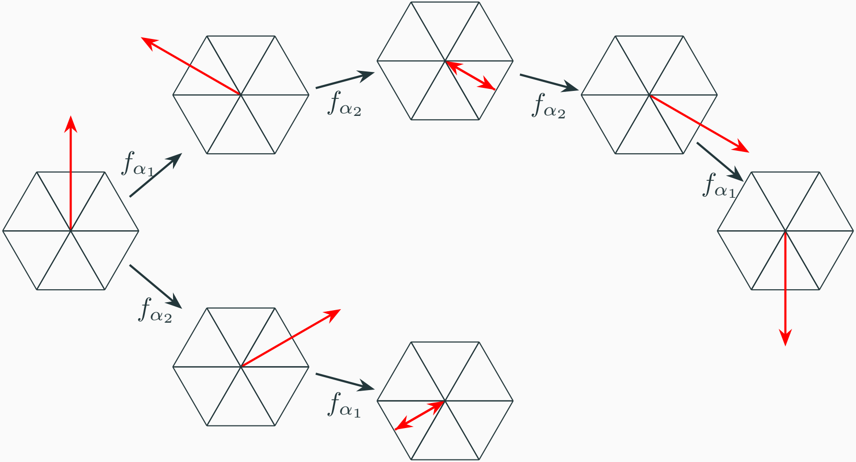

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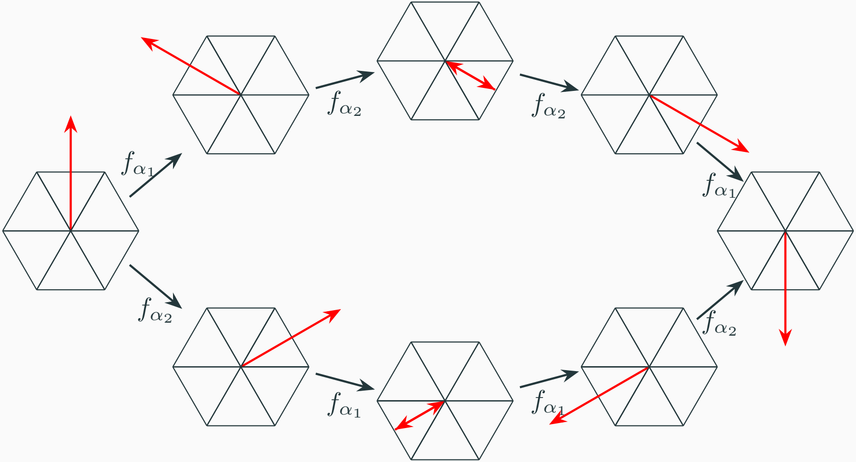

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Notation: $\mathcal{A}\eta$ loop model generated from η

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Bott-Samelson manifolds

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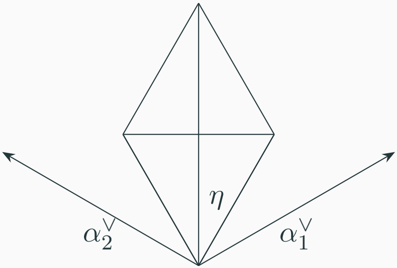

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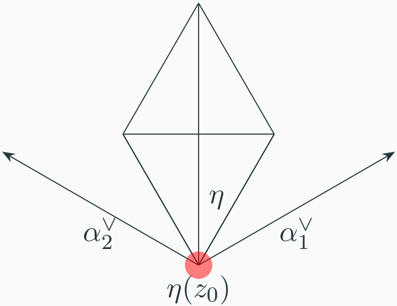

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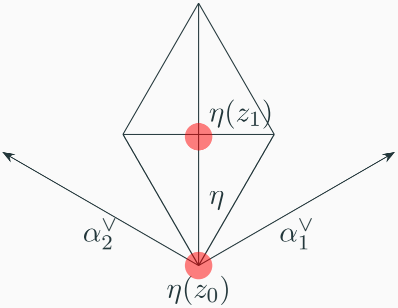

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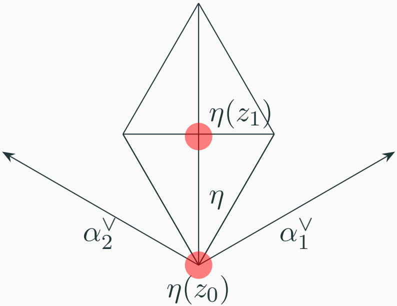

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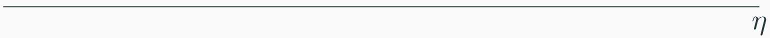

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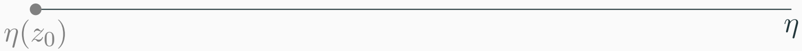

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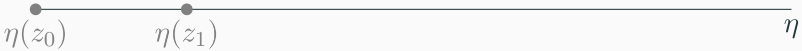

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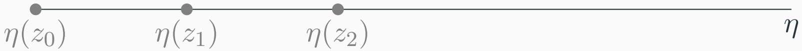

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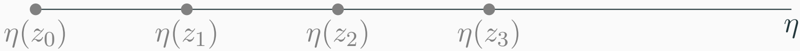

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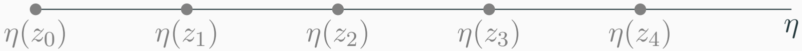

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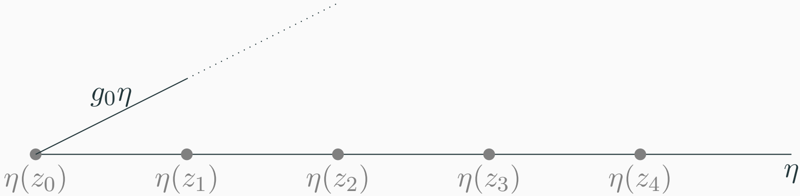

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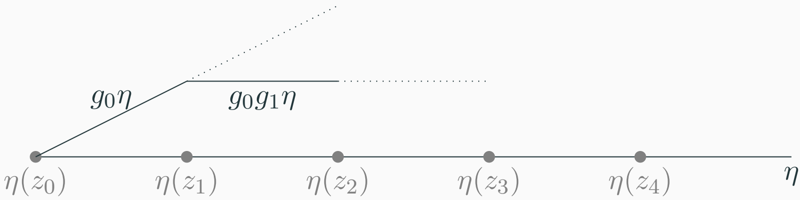

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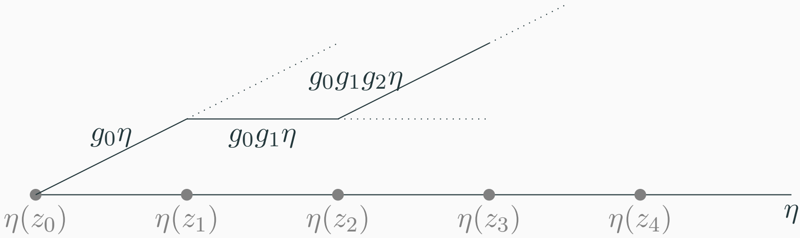

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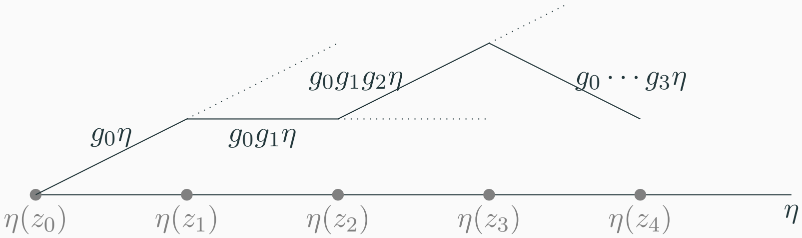

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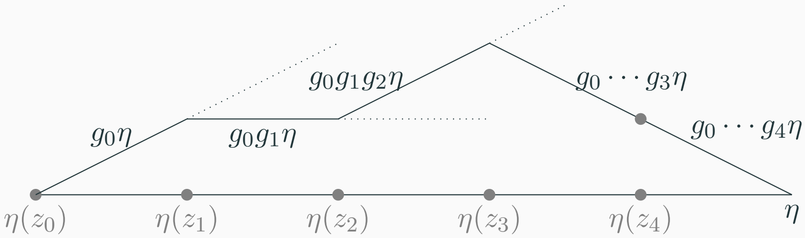

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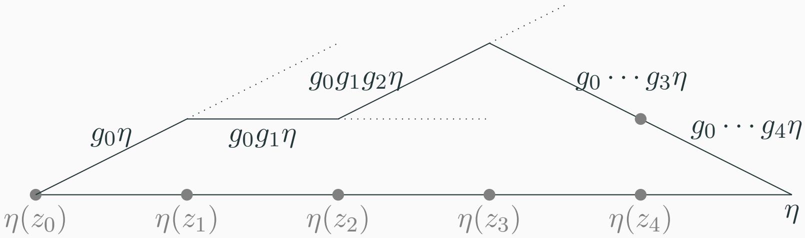

- smooth and S-equivariant embedding $\Gamma_\eta \to \Omega(K)$ via $f_\eta([g_0, \ldots, g_t]) = [g_0, \ldots, g_t].\eta$.

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- Squint $\implies$ coadjoint orbit setting

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Galleries

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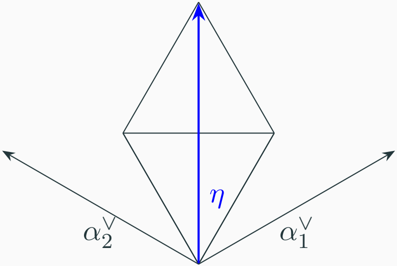
- $\eta \in \Omega(S) \rightsquigarrow$ generalized gallery $\delta(\eta)$.



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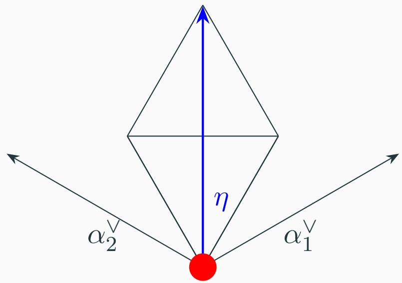

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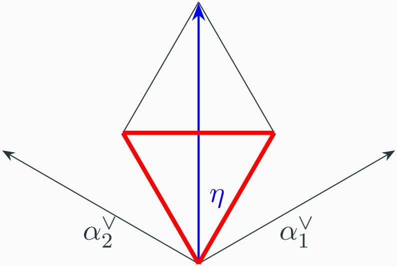

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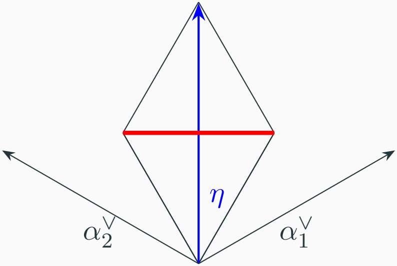

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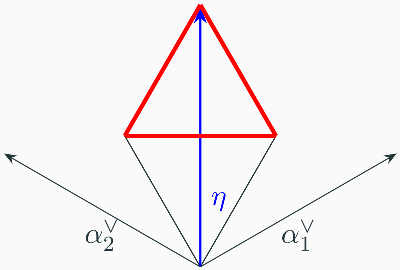

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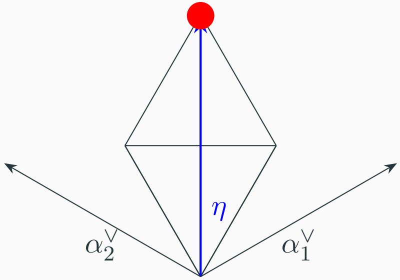

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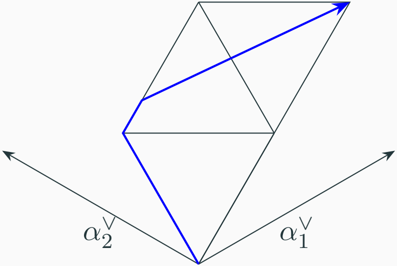

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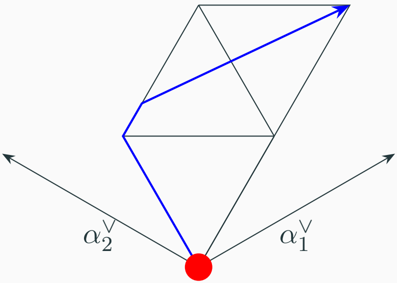

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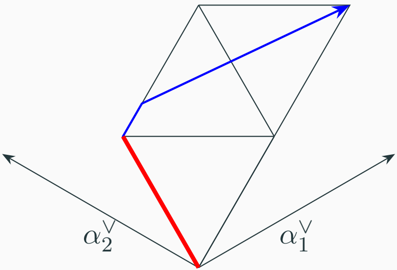

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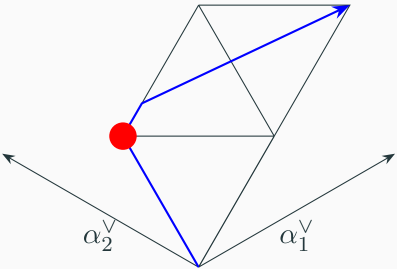

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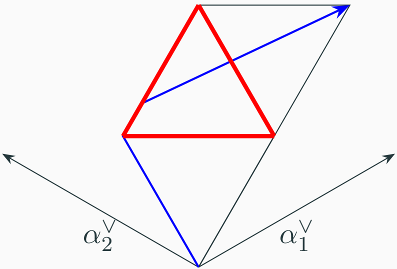

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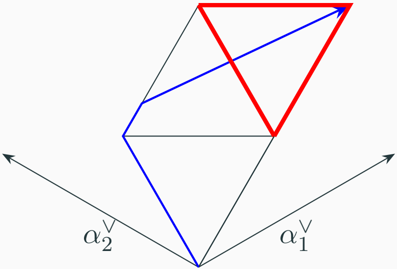

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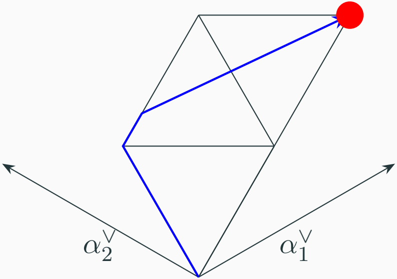

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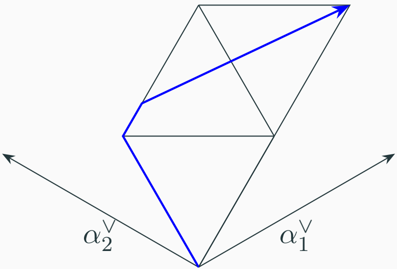

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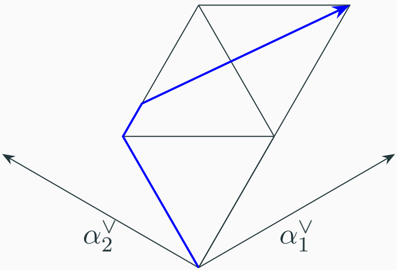

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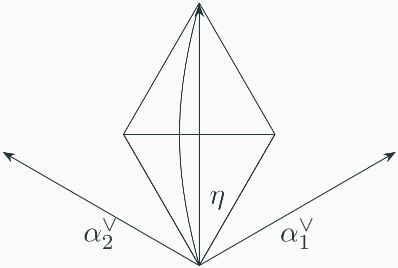

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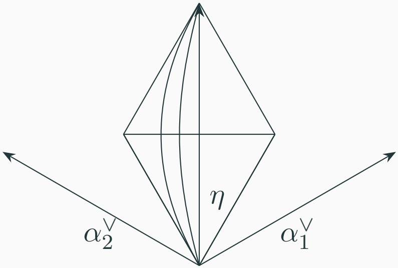

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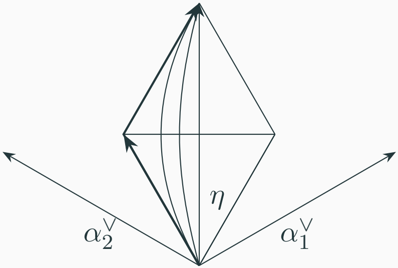

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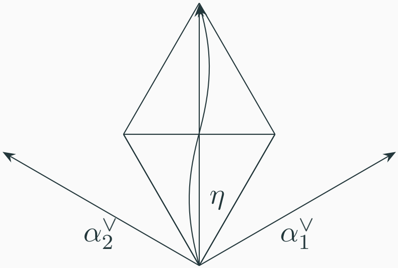
$$\varphi = f_1^{-1} \circ \lim_{s \to 1} f_s$$



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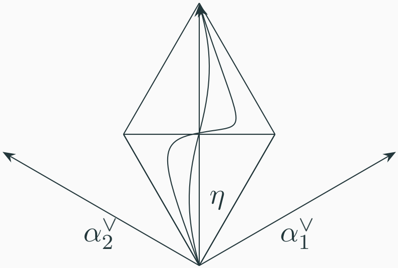

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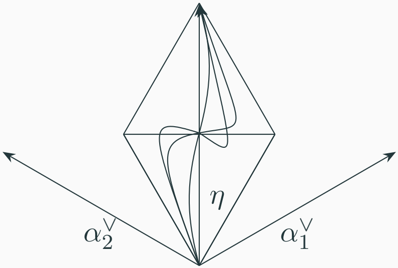

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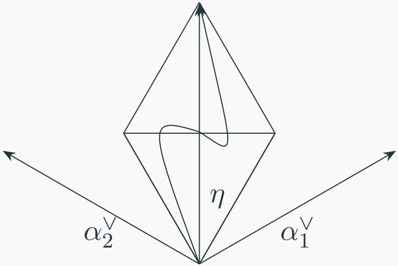

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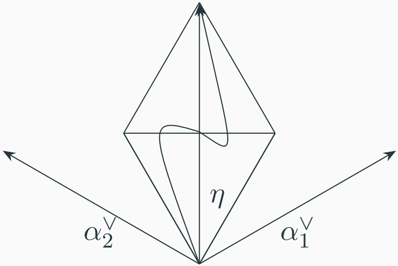

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- $\implies$ *new class of loops with* $\mu(\Gamma_\eta)$ *Weyl polytope*

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- $\Omega(S) \subseteq \Omega(\mathrm{SU}_n)$ non-discrete

Smearing the affine Schubert variety

For simplicity K simply connected

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Image diffeomorphic to affine Schubert variety
Induced embedding isotopic to inclusion map, also for MV cycles.

Results

- ✓ Root operators descend to loop group of compact torus.
- ✓ Loop model embeds into generalized Bott–Samelson manifold Γ_η
- ✓ Γ_η symplectic for η in dominant direction
- ✓ Moment Polytope is Weyl polytope
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  New embeddings of affine Schubert variety and MV cycles into loop group

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The road ahead

More Work

Where to go from here

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- Extend root operators to $\Omega(K)$, Birkhoff decomposition
- $\bigcup_\nu \mathrm{Im}(\pi_\nu)$ what is this space?

Thanks for your attention, and stay healthy

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